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The Dynamics of three species food web model with disease in the intermediate predator

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Abstract:

An eco-epidemiological model consisting of a three-species food web model, with an *SIS* **epidemic disease in the intermediate predator, is proposed and analyzed. It is assumed that the disease transmitted between the individual of intermediate predator species only through an external factors as well as contact. The existence, uniqueness and boundedness of the solution of the system are studied. The existence of all possible equilibrium points are discussed. The local as well as global stability analysis of each equilibrium point is investigated. Finally further investigations for the global dynamics of the proposed system are carried out with the help of numerical simulations. It is observed that the system has one type of attractors, its approaches asymptotically to one of its equilibrium points.**

Keywords: Food web model, SIS epidemic disease, Stability, Lyapunov function.

1. Introduction:

The food web is one important approach to the study of an ecological community. The theoretical studies of food webs must contend with the question of how to couple the large number of interacting species so that all the species persist for the forward time. One line of investigation assumes that the "building blocks" are species interacting in pair wise fashion [7-8]. Behavior of the entire community is then assumed to arise from the coupling of these strongly interacting pairs. This approach is tractable to theoretical analysis. Moreover, the considerable intuition with two-species models may be applied to community food web questions. The critical behavior to community function may arise only through the interaction of three or more species. In fact, the behaviors seen in community models involving three or more species are much more complicated than those seen in continuous two species ecological models [9-11,15-26]. On the other hand an epidemiological systems have been extensively studied in literatures, see for example [12-14] and the reference there in.

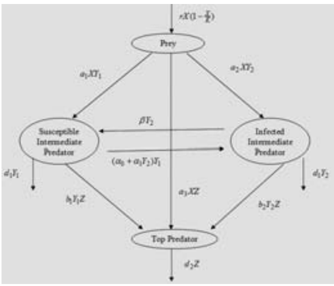
It is well know that, in nature species does not exist alone. In fact, any given habitat may contain dozens or hundreds of species, sometimes thousands. Consequently, the possibility of spread of the disease in a community becomes larger as the number of infected species in the habitat increases. Accordingly, the study of the effect of disease on the dynamical behavior of interacting species has a vital biological significance in ecology. Keeping the above in view, ecology and epidemiology are major and very interested fields for the study of their own right. But there are some common feature between these systems. It is interesting and important from biological point of view to study ecological systems under the influence of epidemiological factors. Quite a good number of studies have already been performed in eco-epidemiological systems[1-5]. Anderson and May(1982)[6] who were the first scientists which proposed an eco-epidemiological model by merging the Lotka-Volttera prey-predator model and Kermack and Mckendrick *SIR* epidemiological model. Most of the studies on eco-epidemiological systems are based on prey-dependent model with disease in the prey population. In this papper, a three-species food web model with an *SIS* disease in intermediate predator is proposed and

2. The mathematical model

Consider an eco-epidemiological model consisting of a **prey** species $X(T)$ that denotes to population density of prey at time T ; **intermediate predator** that divided into two classes: susceptible $Y_1(T)$ and infected $Y_2(T)$, here $Y_1(T)$ and $Y_2(T)$ represent the population density at time T for the susceptible and infected intermediate predator respectively; and the **top predator** species that assigned to its population density at time T by $Z(T)$. Now in order to formulates the above model mathematically the following assumptions are considered:

1. The prey $X(T)$ grows logistically in the absence of predation with intrinsic growth rate $r>0$ and carrying capacity $K>0$.
2. The predators $Y_1(T)$, $Y_2(T)$ and $Z(T)$ consume the prey $X(T)$ according to Lotka-Vollterra functional responses with attack rates $a_1>0$, $a_2>0$, $a_3>0$ and conversion rates $e_1>0$, $e_2>0$, $e_3>0$ respectively. Further, it is assumed that the top predator $Z(T)$ also preys upon the susceptible intermediate predator $Y_1(T)$ and infected intermediate predator $Y_2(T)$ with attack rates $b_1>0$, $b_2>0$ and conversion rates $e_4>0$, $e_5>0$ respectively.
3. The disease transmitted within the population of intermediate predator only by contact between the intermediate predator individuals, according to simple mass action law with contact infected rate $\alpha_1>0$, and through an external source for disease with an external infected rate $\alpha_0>0$.
4. The infected individuals of intermediate predator $Y_2(T)$ is recovered and they become susceptible $Y_1(T)$ again with recover rate constant $\beta>0$.
5. It is assumed that the disease does not caused death instead there is natural death rates only represented by $d_1>0$ and $d_2>0$ for the intermediate predator and top predator respectively.

Accordingly, the path interactions between the species in the above model can be illustrated in the following block diagram.



Blok diagram of our proposed model.

Moreover the dynamics of the above model can be represented by the following set of nonlinear first order differential equations:

$$
\begin{aligned}
\frac{dX}{dT} &= rX\left(1-\frac{X}{K}\right)-a_1XY_1-a_2XY_2-a_3XZ\,,\,X(0)\geq 0\\
\frac{dY_1}{dT} &= e_1a_1XY_1-b_1Y_1Z-(\alpha_0+\alpha_1Y_2)Y_1+\beta Y_2-d_1Y_1,Y_1(0)\geq 0\\
\frac{dY_2}{dT} &= e_2a_2XY_2-b_2Y_2Z+(\alpha_0+\alpha_1Y_2)Y_1-\beta Y_2-d_1Y_2,Y_2(0)\geq 0\\
\frac{dZ}{dT} &= e_3a_3XZ+e_4b_1Y_1Z+e_5b_2Y_2Z-d_2Z,Z(0)\geq 0
\end{aligned}
\tag{1}
$$

Clearly, system (1) is a three-species food web model involving an *SIS* epidemic disease within the population of the intermediate predator only.

Note that the above model contains 16 positive parameters in all, which makes mathematical analysis of the system very difficult. So in order to reduce the number of parameters and determined which parameter represents the control parameter, the following dimensionless variable are used:

$$t = rT,\ x = \frac{X}{K},\ y_1 = \frac{a_1 Y_1}{r},\ y_2 = \frac{a_2 Y_2}{r},\ z = \frac{a_3 Z}{r}$$

Accordingly, system (1) can be rewritten in the following non dimensional form:

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x) - xy_1 - xy_2 - xz = f_1(x, y_1, y_2, z) \\
\frac{dy_1}{dt} &= r_1 xy_1 - r_2 y_1 z - (r_3 + r_4 y_2) y_1 + r_5 y_2 - r_6 y_1 = f_2(x, y_1, y_2, z) \\
\frac{dy_2}{dt} &= r_7 xy_2 - r_8 y_2 z + r_9 (r_3 + r_4 y_2) y_1 - r_{10} y_2 - r_6 y_2 = f_3(x, y_1, y_2, z) \\
\frac{dz}{dt} &= r_{11} xz + r_{12} y_1 z + r_{13} y_2 z - r_{14} z = f_4(x, y_1, y_2, z)
\end{aligned} \tag{2}$$

here $x(0) \geq 0$, $y_1(0) \geq 0$, $y_2(0) \geq 0$ and $z(0) \geq 0$ with the following constants represent the non dimensional parameters

$$r_1 = \frac{e_1 a_1 K}{r},\quad r_5 = \frac{\beta a_1}{r a_2},\quad r_9 = \frac{a_2}{a_1}\quad r_2 = \frac{b_1}{a_3},$$
$$r_6 = \frac{d_1}{r},\quad r_{10} = \frac{\beta}{r},\quad r_3 = \frac{\alpha_0}{r},\quad r_7 = \frac{e_2 a_2 K}{r}$$
$$r_{11} = \frac{e_3 a_3 K}{r},\ r_4 = \frac{\alpha_1}{a_2},\ r_8 = \frac{b_2}{a_3},\ r_{12} = \frac{e_4 b_1}{a_1},$$
$$r_{13} = \frac{e_5 b_2}{a_2} \text{ and } r_{14} = \frac{d_2}{r}$$

It has been observed that the non dimensional system (2) contains 14 parameters only, while the original system (1) contains 16 parameters .

Obviously the interaction functions f_1, f_2, f_3 and f_4 of the system (2) are continuous and have continuous partial derivatives on the state space R_+^4 , therefore these functions are lipschizian on its domain R_+^4 and then the solution of system (2) with non negative initial condition exists and is unique. In addition, all the solutions of system (2) which initiate in R_+^4 are uniformly bounded as shown in the following theorem.

Theorem 1: All solutions of system (2) that initiate in the state space R_+^4 are uniformly bounded in the region $\Psi = \{(x, y_1, y_2, z) \in R_+^4 : 0 \leq ax + r_9 y_1 + y_2 + bz \leq \frac{2a}{c}\}$ where,

$$a = \max\{r_1 r_9, r_7, b r_{11}\}, b = \min\left\{\frac{r_8}{r_{13}}, \frac{r_2 r_9}{r_{12}}\right\}$$

and $c = \min\{1, r_6, r_{14}\}$.

Proof. From the first equation of the model, we have $\frac{dx}{dt} \leq x(1-x)$ then by solving this differential inequality we get $x(t) \leq 1$ as $t \to \infty$. Now let $(x(t), y_1(t), y_2(t), z(t))$ be any solution of the system (2) with the non negative initial conditions. Assume that $R = ax + r_9 y_1 + y_2 + bz$, then from the system (2) equations we get

$$\begin{aligned}
\frac{dR}{dt} &= \leq 2a - c(ax + r_9 y_1 + y_2 + bz) \\
&= 2a - cR
\end{aligned}$$

So again by solving the above linear differential inequality we get that

$R(t) \leq \frac{2a}{c}$ as $t \to \infty$.

The proof is completed.

3. Existence of equilibrium points

It is well known from the biological point of view that, the existence of intermediate predator population independently from its prey population is impossible because the predator individuals can not be survive without the existence of the prey individuals. Therefore, system (2) have at most five nonnegative equilibrium points, which can be described as follows:

The trivial equilibrium point $E_{30} = (0,0,0,0)$ and the axial equilibrium point $E_{31}=(1,0,0,0)$ are always exist. However the intermediate predator free equilibrium point $E_{32}=(\bar{x},0,0,\bar{z})$ $=\left(\frac{r_{14}}{r_{11}},0,0,1-\frac{r_{14}}{r_{11}}\right)$ exists under the condition $r_{14}<r_{11}$.

The top predator free equilibrium point $E_{33}=(\hat{x},\hat{y}_1,\hat{y}_2,0)$ where

$$\hat{x}=1-(\hat{y}_1+\hat{y}_2) \quad ; \quad \hat{y}_1=\frac{C+\frac{r_3r_9}{m}}{D+r_7m} \quad \text{and}$$

$$\hat{y}_2=\frac{A+r_5m}{B+\frac{r_1}{m}}$$

here we have $A=r_1-(r_3+r_6)$, $B=r_1+r_4>0$, $C=r_7-(r_6+r_{10})$ and $D=r_7-r_4r_9$, while m is a positive root for

$$r_5r_7m^3+(r_7A+r_5D-BC)m^2 \\ +(AD-Br_3r_9-r_1C)m \quad -r_1r_3r_9=0$$

Accordingly, E_{33} exists uniquely in the $Int.R_+^3$ of xy_1y_2- space under the following set of conditions:

$$\left.\begin{array}{l} r_7r_5+(r_1+r_4)(r_6+r_{10})>r_7(r_4+r_3+r_6)+r_4r_5r_9, \\ \text{or} \\ r_4r_6r_9+r_1(r_6+r_{10})<r_1r_9(r_3+r_4)+r_7(r_3+r_6) \end{array}\right\}$$

$$r_1+r_5m>r_3+r_6$$

$$\left.\begin{array}{l} r_7+\frac{r_3r_9}{m}>r_6+r_{10} \text{ with } r_7+r_7m>r_4r_9, \\ \text{or} \\ r_7+\frac{r_3r_9}{m}<r_6+r_{10} \text{ with } r_7+r_7m<r_4r_9 \end{array}\right\}$$

$$\hat{y}_1+\hat{y}_2<1$$

Note that it is easy to verify that, conditions guarantees that ($r_7A+r_5D-BC>0$ or $AD-Br_3r_9-r_1C<0$) and $\hat{y}_2>0$, $\hat{y}_1>0$, $\hat{x}>0$.

Finally the positive equilibrium point $E_{34}=(x^*,y_1^*,y_2^*,z^*)$ exists uniquely in the $Int.R_+^4$ if there is a positive solution to the following set of algebraic equations:

$$1-x-y_1-y_2-z=0$$
$$r_1xy_1-r_2y_1z-(r_3+r_4y_2)y_1+r_5y_2-r_6y_1=0$$
$$r_7xy_2-r_8y_2z+r_9(r_3+r_4y_2)y_1-r_{10}y_2-r_6y_2=0$$
$$r_{11}x+r_{12}y_1+r_{13}y_2-r_{14}=0$$

Straightforward computations give that:

$$x=\frac{r_{14}}{r_{11}}-\left(\frac{r_{12}}{r_{11}}y_1+\frac{r_{13}}{r_{11}}y_2\right)$$

$$y_2=-\left(\frac{A_1y_1+C_1}{r_5+B_1y_1}\right)y_1$$

$$z=\left(1-\frac{r_{14}}{r_{11}}\right)+\left(\frac{r_{12}}{r_{11}}-1\right)y_1+\left(\frac{r_{13}}{r_{11}}-1\right)y_2$$

$$A_3{y_1}^3+B_3{y_1}^2+C_3y_1+D_3=0$$

where

$$A_3=A_1(A_1A_2-B_1B_2)$$
$$B_3=2A_1A_2C_1-A_1(r_5B_2+C_2B_1) \\ -B_1(C_1B_2-r_3r_9B_1)$$
$$C_3=A_2{C_1}^2-A_1r_5C_2-C_1(r_5B_2+B_1C_2) \\ +2r_3r_5r_9B_1$$
$$D_3=r_5(r_3r_5r_9-C_1C_2)$$

With

$$A_1=r_2-r_1\frac{r_{12}}{r_{11}}-r_2\frac{r_{12}}{r_{11}} \quad ,$$

$$A_2=r_8-(r_7+r_8)\frac{r_{13}}{r_{11}},$$

$$B_1=r_2-(r_1+r_2)\frac{r_{13}}{r_{11}}-r_4 \quad ,$$

$$B_2=r_8+r_4r_9-(r_7+r_8)\frac{r_{12}}{r_{11}},$$

$$C_1=(r_1+r_2)\frac{r_{14}}{r_{11}}-(r_2+r_3+r_6) \quad ,$$

$$C_2=(r_7+r_8)\frac{r_{14}}{r_{11}}-(r_6+r_8+r_{10})$$

That is if the set of the following sets of conditions holds:

$$A_3>0; B_3>0 \text{ with } D_3<0,$$
$$r_{14}>r_{12}{y_1}^*+r_{13}{y_2}^*$$
$$r_{14}<r_{11}<\min\{r_{12},r_{13}\}$$
$$\frac{A_1y_1+C_1}{r_5+B_1y_1}<0$$

4. Stability analysis of equilibrium points

Now in order to study the local stability of system (2), the Variational matrix of system (2) is computed at each of the above equilibrium points and then the eigenvalues are determined as shown in the following.

The Variational matrix at the trivial equilibrium point E_{30} is given by:

$$V(E_{30})=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -(r_3+r_6) & r_5 & 0\\ 0 & r_3r_9 & -(r_6+r_{10}) & 0\\ 0 & 0 & 0 & -r_{14}\end{pmatrix}$$

So, the eigenvalues are

$$\gamma_{01}=1>0,\ \gamma_{03}=-r_{14}<0$$

$$\gamma_{02},\gamma_{04}=-\tfrac{(r_3+2r_6+r_{10})}{2}\pm \tfrac{1}{2}\sqrt{(r_3+2r_6+r_{10})^2-4[(r_3+r_6)(r_6+r_{10})-r_3r_5r_9]}$$

Therefore E_{30} is a saddle point.

The Variational matrix at the axial equilibrium point E_{31} can be written as

$$V(E_{31})=\begin{pmatrix}-1 & -1 & -1 & -1\\ 0 & r_1-(r_3+r_6) & r_5 & 0\\ 0 & r_3r_9 & r_7-(r_6+r_{10}) & 0\\ 0 & 0 & 0 & r_{11}-r_{14}\end{pmatrix}$$

Then the eigenvalues of $V(E_{31})$ can be written as follows

$$\gamma_{11}=-1<0,\ \lambda_{14}=r_{11}-r_{14}$$

$$\gamma_{12}+\gamma_{13}=A_{11}=r_1-(r_3+r_6)+r_7-(r_6+r_{10})$$

$$\gamma_{12}\gamma_{13}=A_{12}=(r_1-(r_3+r_6))(r_7-(r_6+r_{10}))-r_3r_5r_9$$

Consequently, E_{31} is the local asymptotically stable provided that the following conditions are satisfied:

$r_{11}<r_{14}$

$r_1<r_3+r_6$; $r_7<r_6+r_{10}$

$(r_1-(r_3+r_6))(r_7-(r_6+r_{10}))>r_3r_5r_9$

Otherwise it is saddle point. Furthermore in the following theorem the global stability conditions of E_{31} are established.

Theorem 2: Assume that E_{31} is locally asymptotically stable in R_+^4. Then, it is globally asymptotically stable, provided that the following conditions hold:

$$r_9\le 1$$

$$\operatorname{Max}\{r_7,r_{11}\}\le r_1\le \min\{r_6r_9,r_{14}\}$$

$$r_{12}\le r_2r_9$$

$$r_{13}\le r_8$$

Proof: Consider the following function:

$$L_1(x,y_1,y_2,z)=r_1[x-1-\ln(x)]+r_9y_1+y_2+z$$

it is easy to see that $L_1(x,y_1,y_2,z)\in C^1(R_+^4,R)$, in addition $L_1(1,0,0,0)=0$, while $L_1(x,y_1,y_2,z)>0$; $\forall(x,y_1,y_2,z)\in R_+^4$ and $(x,y_1,y_2,z)\ne(1,0,0,0)$. Further

$$\frac{dL_1}{dt}=-r_1(x-1)^2-r_1(1-r_9)xy_1-(r_6r_9-r_1)y_1$$
$$-(r_1-r_7)xy_2-(r_6-r_1)y_2-(r_1-r_{11})xz$$
$$-(r_2r_9-r_{12})y_1z-(r_8-r_{13})y_2z$$

Now due to the given conditions, it is obtain that $\frac{dL_1}{dt}<0$. Therefore, $\frac{dL_1}{dt}$ is negative definite, and hence the proof is complete.

■

Theorem 3: Assume that the intermediate predator free equilibrium point E_{32} of system (2) exists. Then it is locally asymptotically stable provided that

$$\bar{x}<\min\left\{\left(\frac{r_2}{r_1}\bar{z}+\frac{r_3+r_6}{r_1}\right),\left(\frac{r_8}{r_7}\bar{z}+\frac{r_6+r_{10}}{r_7}\right)\right\}$$

$$\bar{a}\bar{b}>r_3r_5r_9$$

Where

$\bar{a}=r_1\bar{x}-r_2\bar{z}-r_3-r_6$ and $\bar{b}=r_7\bar{x}-r_8\bar{z}-r_{10}-r_6$

Proof. It is easy to verify that the Variational matrix for the system (2) at the point E_{32} is given by

$$V(E_{32})=\begin{pmatrix}-\bar{x} & -\bar{x} & -\bar{x} & -\bar{x}\\ 0 & \bar{a} & r_5 & 0\\ 0 & r_3r_9 & \bar{b} & 0\\ r_{11}\bar{z} & r_{12}\bar{z} & r_{13}\bar{z} & 0\end{pmatrix}$$

Then the characteristic equation corresponding to the equilibrium point E_{32} is given by:

$$\gamma^4 + D_1\gamma^3 + D_2\gamma^2 + D_3\gamma + D_4 = 0$$

here $D_1 = \bar{x} - \bar{A}$, $D_2 = -\bar{x}\bar{A} + \bar{B} + r_{11}\bar{x}\bar{z}$, $D_3 = \bar{x}\bar{B} - r_{11}\bar{x}\bar{z}\bar{A}$, $D_4 = r_{11}\bar{x}\bar{z}\bar{B}$ with $\bar{A} = \bar{a} + \bar{b}$ and $\bar{B} = \bar{a}\bar{b} - r_3 r_5 r_9$. Now straightforward computation shows that, The conditions , guarantee that $\bar{A} < 0$ and $\bar{B} > 0$. Consequently, it obtain $D_i > 0$ for $i = 1,2,3,4$ and that;

$$\begin{aligned}\left(D_1D_2 - D_3\right)D_3 - D_1^2 D_4 &= -\bar{x}^3\bar{A}\bar{B} + \bar{x}^2\bar{A}^2\bar{B} - \bar{x}\bar{A}\bar{B}^2 \\ &\quad + r_{11}\bar{x}^3\bar{z}\bar{A}^2 - r_{11}^2\bar{x}^3\bar{z}^2\bar{A} \\ &\quad + r_{11}\bar{x}^2\bar{z}\bar{A}\left(2\bar{B} - A^2\right) > 0\end{aligned}$$

Thus by Routh-Hurwitz criterion all the eigenvalues of $V(E_{32})$ have negative real parts, that is the intermediate predator free equilibrium point E_{32} is locally asymptotically stable, and hence the proof is complete.

Theorem 4. Assume that the intermediate predator free equilibrium point E_{32} is locally asymptotically stable and let the following inequalities hold.

$$\max\left\{\frac{r_1 r_9}{r_{11}}, \frac{r_7}{r_{11}}\right\} < \min\left\{\frac{r_6 r_9}{r_{11}\bar{x}}, \frac{r_6}{r_{11}\bar{x}}, \frac{r_2 r_9}{r_{12}}, \frac{r_8}{r_{13}}\right\}$$

Then it is globally asymptotically stable in the R_+^4.

Proof. According to condition (13), there exists a positive number c_1, such that,

$$\max\left\{\frac{r_1 r_9}{r_{11}}, \frac{r_7}{r_{11}}\right\} < c_1 < \min\left\{\frac{r_6 r_9}{r_{11}\bar{x}}, \frac{r_6}{r_{11}\bar{x}}, \frac{r_2 r_9}{r_{12}}, \frac{r_8}{r_{13}}\right\}$$

Now, consider the following function:

$$\begin{aligned}L_2(x, y_1, y_2, z) &= r_{11}c_1\left(x - \bar{x} - \bar{x}\ln\left(\frac{x}{\bar{x}}\right)\right) + r_9 y_1 \\ &\quad + y_2 + c_1\left(z - \bar{z} - \bar{z}\ln\left(\frac{z}{\bar{z}}\right)\right)\end{aligned}$$

It is easy to see that $L_2(x, y_1, y_2, z) \in C^1(R_+^4, R)$ and $L_2(\bar{x}, 0, 0, \bar{z}) = 0$, while $L_2(x, y_1, y_2, z) > 0$ for all $(x, y_1, y_2, z) \neq (\bar{x}, 0, 0, \bar{z})$ in R_+^4 and. Now, since straightforward computation leads to:

$$\begin{aligned}\frac{dL_2}{dt} &< -r_{11}c_1\left(x - \bar{x}\right)^2 - (r_{11}c_1 - r_1 r_9)xy_1 \\ &\quad - (r_{11}c_1 - r_7)xy_2 - (r_6 r_9 - r_{11}c_1\bar{x})y_1 \\ &\quad - (r_6 - r_{11}c_1\bar{x})y_2 - (r_2 r_9 - r_{12}c_1)y_1 z \\ &\quad - (r_8 - r_{13}c_1)y_2 z\end{aligned}$$

Consequently, due to conditions, we obtain that: $\frac{dL_2}{dt} < 0$ for all initial point $(x, y_1, y_2, z,) \in R_+^4$. Hence, $\frac{dL_2}{dt}$ is negative definite and then L_2 is a Lyapunov function with respect to E_{32} . So the intermediate predator free equilibrium point E_{32} is globally asymptotically stable in the R_+^4 , which complete the proof.

Theorem 5. Assume that the top predator free equilibrium point $E_{33} = (\hat{x}, \hat{y}_1, \hat{y}_2, 0)$ of system (2) exists. Then it is locally asymptotically stable provided that

$$\hat{x} < \min\left\{\frac{r_4\hat{y}_2 + r_3 + r_6}{r_1}, \frac{r_6 + r_{10} - r_4 r_9\hat{y}_1}{r_7}, \frac{r_{14} - r_{12}\hat{y}_1 - r_{13}\hat{y}_2}{r_{11}}\right\}$$

$$\hat{y}_1 > \frac{r_5}{r_4}$$

$$0 < \hat{y}_2 < \frac{r_3 r_9}{r_2 - r_4 r_9}$$

$$\begin{aligned}\left(\left(r_1 + r_7\right)\hat{x} + r_4 r_9\hat{y}_1 - r_4\hat{y}_2 - r_3 - r_{10} - 2r_6\right)\left(r_5 - r_4\hat{y}_2\right) \\ > r_1\hat{x}\hat{y}_1\end{aligned}$$

Proof. It is easy to verify that the Variational matrix for the system (2) at the point E_{33} is given by:

$$\begin{aligned}V(E_4) &= \begin{pmatrix} -\hat{x} & -\hat{x} & -\hat{x} & -\hat{x} \\ r_1\hat{y}_1 & r_1\hat{x} - r_4\hat{y}_2 - r_3 - r_6 & -r_4\hat{y}_1 + r_5 & -r_2\hat{y}_1 \\ r_2\hat{y}_2 & r_3 r_9 + r_4 r_9\hat{y}_2 & r_7\hat{x} + r_4 r_9\hat{y}_1 - r_6 - r_{10} & -r_8\hat{y}_2 \\ 0 & 0 & 0 & r_{11}\hat{x} + r_{12}\hat{y}_1 + r_{13}\hat{y}_2 - r_{14} \end{pmatrix} \\ &= (b_{ij})_{4\times 4}\end{aligned}$$

Clearly the characteristic equation of the above Variational matrix can be written as

$$\left(\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3\right)\left(b_{44} - \lambda\right) = 0$$

Here

$$B_1 = -(b_{11} + b_{22} + b_{33}) \quad ,$$

$$B_2 = b_{11}b_{22} - b_{12}b_{21} + b_{11}b_{33} - b_{13}b_{31} + b_{22}b_{33} - b_{23}b_{32} \quad \text{and}$$

$$\begin{aligned} B_3 = -b_{11}b_{22}b_{33} - b_{12}b_{23}b_{31} - b_{13}b_{21}b_{32} + b_{13}b_{22}b_{31} \\ + b_{11}b_{23}b_{32} + b_{12}b_{21}b_{33} \end{aligned}.$$

Accordingly, either

$$b_{44} - \lambda = 0$$

which gives the eigenvalue in the fourth direction (i.e. $z-$ direction) and then $\lambda_z = b_{44}$, or we have

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0$$

Now straightforward computation shows that, conditions , guarantee that $b_{22} < 0$, $b_{33} < 0$ and $b_{44} < 0$, $b_{23} < 0$, $b_{11}b_{32} - b_{12}b_{31} < 0$, $\lambda_z < 0$, $B_1 > 0$, $B_3 > 0$ and $\left(b_{22} + b_{33}\right)b_{23} + b_{13}b_{21} > 0$, hence

$$\begin{aligned} B_1B_2 - B_3 = (b_{11} + b_{22})(b_{12}b_{21} - b_{11}b_{22}) \\ + (b_{11} + b_{33})(b_{13}b_{31} - b_{11}b_{33}) \\ - b_{22}b_{33}(2b_{11} + b_{22} + b_{33}) + b_{12}b_{23}b_{31} \\ + \left[\left(b_{22} + b_{33}\right)b_{23} + b_{13}b_{21}\right]b_{32} > 0 \end{aligned}$$

Thus by Routh-Hurwitz criterion all the eigenvalues of $V(E_{33})$ have negative real parts, so the top predator free equilibrium point $E_{33} = (\hat{x}, \hat{y}_1, \hat{y}_2, 0)$ is locally asymptotically stable, and hence the proof is complete

Theorem 6. Assume that the top predator free equilibrium point $E_{33} = \left(\hat{x}, \hat{y}_1, \hat{y}_2, 0\right)$ is locally asymptotically stable in R_+^4. Then it is a globally asymptotically stable in the interior of the region $\Gamma \subseteq R_+^4$, if the following condition holds

$$\frac{r_2}{r_1r_{14}}\hat{y}_1 + \frac{r_8}{r_7r_{14}}\hat{y}_2 + \frac{1}{r_{14}}\hat{x} < \min\left\{\frac{1}{r_{11}}, \frac{r_2}{r_1r_{12}}, \frac{r_8}{r_7r_{13}}\right\}$$

$$\left[\frac{\left(r_5 - r_4\hat{y}_1\right)}{r_1\hat{y}_1} + \frac{\left(r_3r_9 + r_4r_9\hat{y}_2\right)}{r_7\hat{y}_2}\right]^2 < 4\left(\frac{r_4y_2 + r_3 + r_6 - r_1x}{r_1\hat{y}_1}\right)\left(\frac{r_6 + r_{10} - r_7x - r_4r_9y_1}{r_7\hat{y}_2}\right)$$

where

$$\Gamma = \left\{(x, y_1, y_2, z) \in R_+^4 : x < \min.\left\{\frac{r_3+r_6+r_4y_2}{r_1}, \frac{r_6+r_{10}-r_4r_9y_1}{r_7}\right\}\right\}$$

Proof: Consider the following function:

$$L_3(x, y_1, y_2, z) = t_1\left(x - \hat{x} - \hat{x}\ln\left(\frac{x}{\hat{x}}\right)\right) + \frac{t_2}{2}\left(y_1 - \hat{y}_1\right)^2 + \frac{t_3}{2}\left(y_2 - \hat{y}_2\right)^2 + t_4z$$

where t_1, t_2, t_3 and t_4 are positive constants to be determined.

It is easy to see that $L_3(x, y_1, y_2, z) \in C^1(R_+^4, R)$, in addition $L_3(\hat{x}, \hat{y}_1, \hat{y}_2, 0) = 0$, while $L_3(x, y_1, y_2, z) > 0$; $\forall(x, y_1, y_2, z) \in R_+^4$ and $(x, y_1, y_2, z) \neq (\hat{x}, \hat{y}_1, \hat{y}_2, 0)$. Further, since

$$\begin{aligned} \frac{dL_3}{dt} = -t_1\left(x - \hat{x}\right)^2 + \left(t_2r_1\hat{y}_1 - t_1\right)\left(x - \hat{x}\right)\left(y_1 - \hat{y}_1\right) \\ + \left(t_3r_7\hat{y}_2 - t_1\right)\left(x - \hat{x}\right)\left(y_2 - \hat{y}_2\right) \\ + \left(t_4r_{11} - t_1\right)zx + (t_2r_2\hat{y}_1^2 + t_3r_8\hat{y}_2^2 + t_1\hat{x} - r_{14}t_4)z \\ + \left(t_4r_{12} - t_2r_2\hat{y}_1\right)y_1z - t_2\left(r_4y_2 + r_3 + r_6 - r_1x\right)\left(y_1 - \hat{y}_1\right)^2 \\ + \left[t_2\left(r_5 - r_4\hat{y}_1\right) + t_3\left(r_3r_9 + r_4r_9\hat{y}_2\right)\right]\left(y_1 - \hat{y}_1\right)\left(y_2 - \hat{y}_2\right) \\ - t_3\left(r_6 + r_{10} - r_7x - r_4r_9y_1\right)\left(y_2 - \hat{y}_2\right)^2 \\ + \left(t_4r_{13} - t_3r_8\hat{y}_2\right)y_2z \end{aligned}$$

So, by choosing $t_1 = 1$, $t_2 = \frac{1}{r_1\hat{y}_1}$, $t_3 = \frac{1}{r_7\hat{y}_2}$, $t_4 = \frac{r_2\hat{y}_1}{r_1r_{14}} + \frac{r_8\hat{y}_2}{r_7r_{14}} + \frac{\hat{x}}{r_{14}}$, and then substituting them in the above equation with using condition we obtain that:

$$\begin{aligned} \frac{dL_3}{dt} \leq -\left(x - \hat{x}\right)^2 - \frac{\left(r_4y_2 + r_3 + r_6 - r_1x\right)}{r_1\hat{y}_1}\left(y_1 - \hat{y}_1\right)^2 \\ + \left[\frac{\left(r_5 - r_4\hat{y}_1\right)}{r_1\hat{y}_1} + \frac{\left(r_3r_9 + r_4r_9\hat{y}_2\right)}{r_7\hat{y}_2}\right]\left(y_1 - \hat{y}_1\right)\left(y_2 - \hat{y}_2\right) \\ - \frac{\left(r_6 + r_{10} - r_7x - r_4r_9y_1\right)}{r_7\hat{y}_2}\left(y_2 - \hat{y}_2\right)^2 \end{aligned}$$

Moreover by using conditions, we obtain that

$$\begin{aligned} \frac{dL_3}{dt} \leq -\left(x - \hat{x}\right)^2 - \left[\sqrt{\frac{\left(r_4y_2 + r_3 + r_6 - r_1x\right)}{r_1\hat{y}_1}}\left(y_1 - \hat{y}_1\right)\right. \\ \left. - \sqrt{\frac{\left(r_6 + r_{10} - r_7x - r_4r_9y_1\right)}{r_7\hat{y}_2}}\left(y_2 - \hat{y}_2\right)\right]^2 \end{aligned}$$

Clearly we have , $\frac{dL_3}{dt}$ is negative definite in the interior of Γ , and hence the proof is complete.

Theorem 7. Assume that the positive equilibrium point $E_{34}=(x^*,y_1^*,y_2^*,z^*)$ of system (2) exists and let the following inequalities hold:

$$x^* < \min.\left\{\frac{r_2z^*+r_4y_2^*+r_3+r_6}{r_1},\frac{r_8z^*-r_4r_9y_1^*+r_6+r_{10}}{r_7}\right\}$$

$$q_{12}{}^2 < q_{11}q_{22}$$
$$q_{13}{}^2 < q_{11}q_{33}$$
$$q_{23}{}^2 < q_{22}q_{33}$$

here we have:

$$q_{11}=r_{11} \qquad q_{12}=\frac{r_1r_{12}}{r_2}-r_{11} \qquad q_{13}=\frac{r_7r_{13}}{r_8}-r_{11} \quad ;$$

$$q_{22}=\frac{r_{12}}{r_2y_1^*}\left(-r_1x^*+r_2z^*+r_4y_2^*+r_3+r_6\right) \quad ;$$

$$q_{23}=r_4\left(\frac{r_{13}}{r_8}-\frac{r_{12}}{r_2}\right)+\frac{r_5r_{12}}{r_2y_1^*}+\frac{r_3r_9r_{13}}{r_8y_2^*} \quad ; \quad \text{and}$$

$$q_{33}=\frac{r_{13}}{r_8y_2^*}\left(-r_7x^*+r_8z^*-r_4r_9y_1^*+r_6+r_{10}\right)$$

Then it is locally asymptotically stable in the $Int.R_+^4$

Proof. It is easy to verify that, the linearized system of system (2) can be written as

$$\frac{dX}{dt}=\frac{dU}{dt}=V(E_{34})\,U$$

here $X=(x,y_1,y_2,z)^T$ and $U=(u_1,u_2,u_3,u_4)^T$ with $u_1=x-x^*$, $u_2=y_1-y_1^*$, $u_3=y-y_2^*$ and $u_4=z-z^*$. Moreover, $V(E_{34})=(c_{ij})_{4\times4}$; $i,j=1,2,\ldots,4$ is the Variational matrix of system (2) at E_{34} in which

$c_{11}=-x^*$, $c_{12}=-x^*$ $c_{13}=-x^*$ $c_{14}=-x^*$, $c_{21}=r_1y_1^*$, $c_{22}=r_1x^*-r_2z^*-r_4y_2^*-r_3-r_6$, $c_{23}=-r_4y_1^*+r_5$, $c_{24}=-r_2y_1^*$, $c_{31}=r_7y_2^*$, $c_{32}=r_9\left(r_3+r_4y_2^*\right)$ $c_{33}=r_7x^*-r_8z^*+r_4r_9y_1^*-r_{10}-r_6$, $c_{34}=-r_8y_2^*$, $c_{41}=r_{11}z^*$, $c_{42}=r_{12}z^*$ $c_{43}=r_{13}z^*$ and $c_{44}=0$

Now, consider the following function

$$L_4=\frac{r_{11}u_1^2}{2x^*}+\frac{r_{12}u_2^2}{2r_2y_1^*}+\frac{r_{13}u_3^2}{2r_8y_2^*}+\frac{u_4^2}{2z^*}$$

It is clearly that $L_4:R_+^4\to R$ is a continuously differentiable function that satisfy that $L_4(0,0,0,0)=0$ and $L_4(u_1,u_2,u_3,u_4)\neq0$ for all $(u_1,u_2,u_3,u_4)\neq(0,0,0,0)$

Hence L_4 is a positive definite function. Now, by differentiating L_4 with respect to time t, gives

$$\frac{dL_4}{dt}=\frac{r_{11}u_1}{x^*}\frac{du_1}{dt}+\frac{r_{12}u_2}{r_2y_1^*}\frac{du_2}{dt}+\frac{r_{13}u_3}{r_8y_2^*}\frac{du_3}{dt}+\frac{u_4}{z^*}\frac{du_4}{dt}$$

Substituting the values of $\frac{du_i}{dt};i=1,2,3,4$ in the above equation, and after doing some algebraic manipulation; we get that:

$$\frac{dL_4}{dt}=-\frac{q_{11}}{2}u_1{}^2+q_{12}\,u_1u_2-\frac{q_{22}}{2}u_2{}^2-\frac{q_{11}}{2}u_1{}^2$$
$$+q_{13}u_1u_3-\frac{q_{33}}{2}u_3{}^2-\frac{q_{22}}{2}u_2{}^2+q_{23}u_2u_3-\frac{q_{33}}{2}u_3{}^2$$

Obviously, due to conditions, we get that

$$\frac{dL_4}{dt}<-\left[\sqrt{\frac{q_{11}}{2}}u_1-\sqrt{\frac{q_{22}}{2}}u_2\right]^2-\left[\sqrt{\frac{q_{11}}{2}}u_1-\sqrt{\frac{q_{33}}{2}}u_3\right]^2$$
$$-\left[\sqrt{\frac{q_{22}}{2}}u_2-\sqrt{\frac{q_{33}}{2}}u_3\right]^2$$

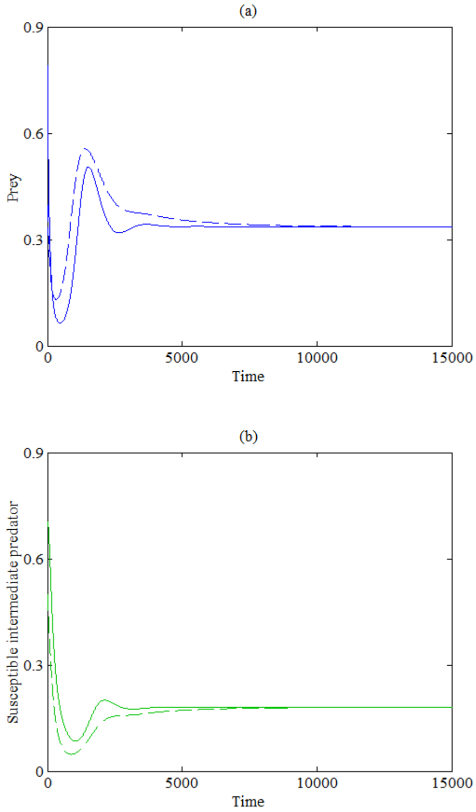
Clearly $\frac{dL_4}{dt}<0$, therefore the origin and then E_{34} is locally asymptotically stable point in the $Int.R_+^4$ and hence the proof is complete.

4.5 Numerical Simulations

In this section the global dynamics of system (2) is investigated numerically. The objectives are confirm our analytical results and discuss the role of the existence of disease in the intermediate predator population on the dynamical behaviour of the system. For the following set of hypothetical, biologically feasible, set of parameters, definitely different set of hypothetical parameters can be chosen also, system (2) is solved numerically starting at different initial points as illustrated in Fig. (1a)-(1d).

$$r_1 = 0.75,\ \ r_2 = 0.5,\ \ r_3 = 0.2,\ \ r_4 = 0.5,\ \ r_5 = 0.2,\ \ r_6 = 0.($$
$$r_7 = 0.5, r_8 = 0.6, r_9 = 0.75, r_{10} = 0.15, r_{11} = 0.3, r_{12} = 0.2$$
$$r_{13} = 0.25, r_{14} = 0.2 \qquad (3)$$

Note that from now onward we will use blue color to describes the trajectory of the prey x ; the green color to describes the trajectory of susceptible intermediate predator y_1 ; the red color to describes the trajectory of infected intermediate predator y_2 ; the sky blue color to describes the trajectory of top predator z ;



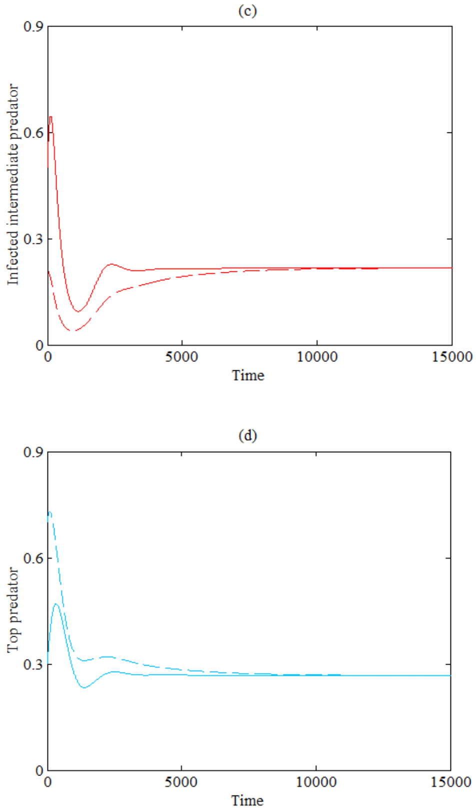


Fig. 1: The solution of system (2) approaches asymptotically to the positive equilibrium point $E_{34} = (0.33, 0.18, 0.21, 0.26)$ for that data given by Eq. (3) starting from two different initial points $(0.8, 0.7, 0.5, 0.3)$ and $(0.4, 0.5, 0.2, 0.7)$ for sold line and dashed line respectively. (a) Trajectories of x . (b) Trajectories of y_1. (c) Trajectories of y_2. (d) Trajectories of z .

Obviously, as shown in Fig. (1), system (2) has a globally asymptotically stable positive equilibrium point in the $Int.R_+^4$ for the data given by Eq. (3). However for the following set of data, system (2) approaches asymptotically to the axial equilibrium point $E_{31} = (1,0,0,0)$ as shown in Fig. (2).

$$r_1 = 0.06,\ \ r_2 = 0.5,\ \ r_3 = 0.2,\ \ r_4 = 0.5,\ \ r_5 = 0.2,$$
$$r_6 = 0.6,\ \ r_7 = 0.05, r_8 = 0.6, r_9 = 0.1, r_{10} = 0.02,$$
$$r_{11} = 0.03, r_{12} = 0.025, r_{13} = 0.025, r_{14} = 0.5 \qquad (4)$$

Clearly, the above set of data satisfy the stability conditions of the axial equilibrium point $E_{31} = (1,0,0,0)$.

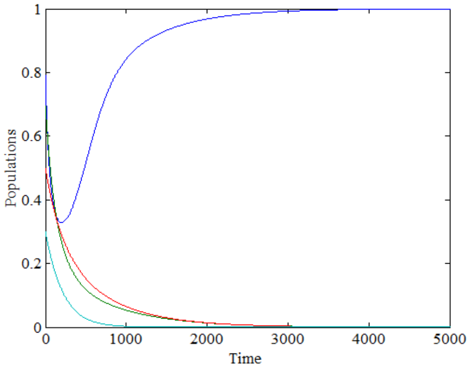


Fig. 2: The solution of system (2) approaches asymptotically to the axial equilibrium point $E_{31} = (1,0,0,0)$ for that data given by Eq. (4).

Accordingly, the solution of system (2) given in Fig. (2) confirm our analytical result in section (4.4). Now further analysis has been down to investigate the effects of varying one parameter each time keeping the rest of parameters as given in Eq. (3) on the dynamical behavior of system (2) in the $Int.R_{+}^{4}$ and the obtained results are given below.

The effect of varying the attack rate of the top predator z to the susceptible intermediate predator y_1 in the range $0 < r_2 < 2.06$ keeping other parameters as given in Eq. (3) is studied. It is observed that system (2) still has a stable positive equilibrium point in the $Int.R_{+}^{4}$, however increasing this parameter further causes extinction in the intermediate predator and the system will approach to the intermediate predator free equilibrium point as shown in the following typical figure.

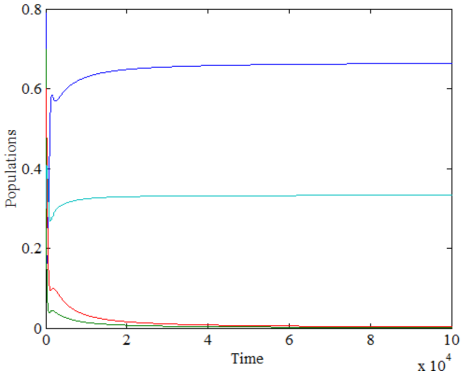


Fig. 3: The solution of system (2) approaches asymptotically to the intermediate predator free equilibrium point $E_{32} = (0.66,0,0,0.33)$ for that data given by Eq. (3) with $r_2 = 2.1$.

The effect of varying the death rate of the intermediate predators (susceptible and infected) in the range $0 < r_6 \leq 0.24$ keeping other parameters as given in Eq. (3) is studied. it is observed that system (2) still has a stable positive equilibrium point in the $Int.R_{+}^{4}$, however increasing this parameter further causes extinction in the intermediate predators (susceptible and infected) and the system will approach to the intermediate predator free equilibrium point as shown in the following typical figure.

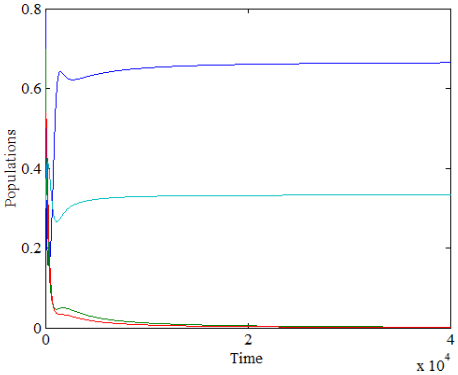


Fig. 4: The solution of system (2) approaches asymptotically to the intermediate predator free equilibrium point $E_{32} = (0.66,0,0,0.33)$ for that data given by Eq. (3) with $r_6 = 0.25$.

Now, the effect of varying the parameter r_{11} , which stand for the conversion rate of the top predator from prey, in the range $0 < r_{11} \leq 0.39$ keeping other parameters as given in Eq. (3) is also studied and the following result is obtained. System (2) has a stable positive equilibrium point in the $Int.R_+^4$, however increasing this parameter further causes extinction in the intermediate predators (susceptible and infected) and the system will approach to the intermediate predator free equilibrium point as shown in the following typical figure

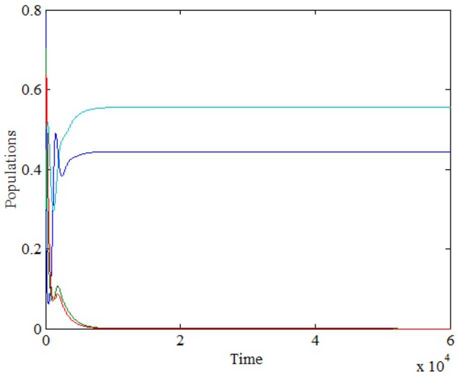


Fig. 5: The solution of system (2) approaches asymptotically to the intermediate predator free equilibrium point $E_{32} = (0.44,0,0,0.55)$ for that data given by Eq. (3) with $r_{11} = 0.45$.

An investigation to the effect of varying the parameter r_{12} , which stand for the conversion rate of the top predator from susceptible intermediate predator, in the range $0.04 \leq r_{12}$ keeping other parameters as given in Eq. (3) is down and the following result is observed. System (2) has a stable positive equilibrium point in the $Int.R_+^4$, however decreasing this parameter further ($r_{12} \leq 0.03$) causes extinction in the top predator and the system will approach to the top predator free equilibrium point as shown in the following typical figure.

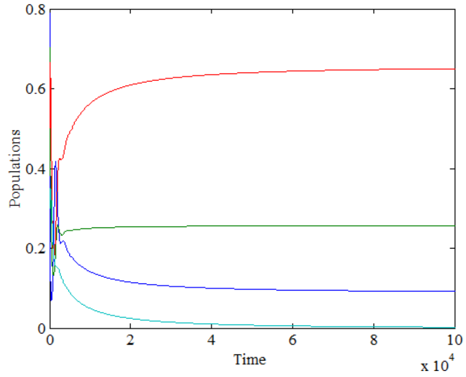


Fig. 6: The solution of system (2) approaches asymptotically to the top predator free equilibrium point $E_{33} = (0.09,0.25,0.65,0)$ for that data given by Eq. (3) with $r_{12} = 0.03$.

Similarly, an investigation to the effect of varying the parameter r_{13} , which stand for the conversion rate of the top predator from infected intermediate predator, in the range $0.17 \leq r_{13}$ keeping other parameters as given in Eq. (3) is down and the following result is observed. System (2) still has a stable positive equilibrium point in the $Int.R_+^4$, however decreasing this parameter further ($r_{13} \leq 0.16$) causes extinction in the top predator and the system will approach to the top predator free equilibrium point as shown in the following typical figure.

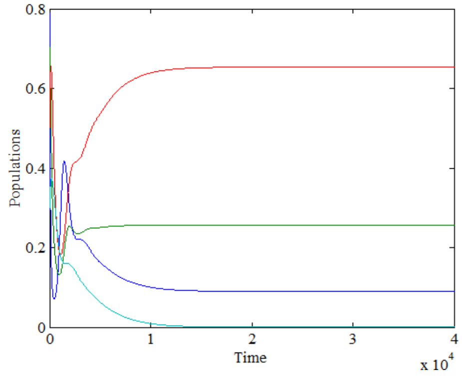


Fig. 7: The solution of system (2) approaches asymptotically to the top predator free equilibrium point $E_{33} = (0.09,0.25,0.65,0)$ for that data given by Eq. (3) with $r_{13} = 0.15$.

Finally, the of varying the death rate of the top predator in the range $0.16 \le r_{14} \le 0.25$ keeping other parameters as given in Eq. (3) is studied. it is observed that system (2) still has a stable positive equilibrium point in the $Int.R_+^4$, however decreasing this parameter further ($r_{14} < 0.16$) causes extinction in the intermediate predators (susceptible and infected) and the system will approach to the intermediate predator free equilibrium point as shown in the typical Fig. (8a), while increasing this parameter further ($r_{14} > 0.25$) causes extinction in the top predator and the system will approach to the top predator free equilibrium point as shown in the typical Fig. (8b).

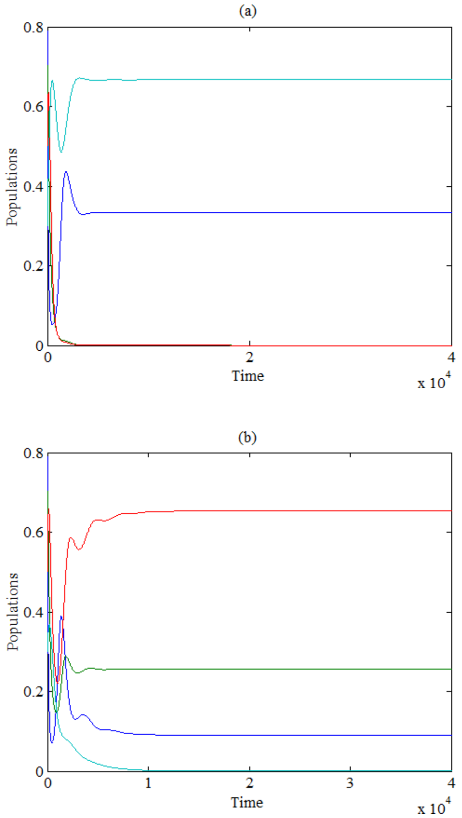


Fig. 8: The solution of system (2) approaches asymptotically to: (a) $E_{32} = (0.33,0,0,0.66)$ for that data given by Eq. (3) with $r_{14} = 0.1$. (b) $E_{33} = (0.09,0.25,0.65,0)$ for that data given by Eq. (3) with $r_{14} = 0.3$

Moreover, it is observed that, varying each of the following parameters (r_1, r_3, r_4, r_5, r_7, r_8 and r_9) with the rest of parameters kept fixed as given by Eq. (3) do not change the dynamical behavior of system (2) in the $Int.R_+^4$, in fact the solution of system (2) still approaches to a positive equilibrium point in the $Int.R_+^4$.

4.6 Discussions and Conclusions:

In this section, a three-species food web model, with an *SIS* epidemic disease in the intermediate predator, is proposed and analyzed. The uniqueness and boundedness of solution of the system are discussed. The existence of all possible equilibrium points are investigated. The local as well as global stability analysis for the proposed system are performed. Moreover, in order to confirm our analytical results and specified which combination of parameters control the dynamical behaviour of system (2) numerical simulations are used for biologically feasible set of hypothetical parameters. For the set of data given by Eq. (3), it is observed that:

1. System (2) has only one type of attractor, approaches to either one of its equilibrium points.
2. System (2) still persists and the solution initiate at any point in the $Int.R_+^4$ approaches asymptotically to the positive equilibrium point for all values of parameters (r_1, r_3, r_4, r_5, r_7, r_8 and r_9) with the rest of parameters kept fixed as given by Eq. (3) .
3. Increasing each of the parameters (r_2, r_6 and r_{11}) with the rest of parameters kept fixed as given by Eq. (3) causes extinction in the intermediate predators (susceptible and infected) and then the system lose the persistence.

4. Decreasing each of the parameters (r_{12} and r_{13}) with the rest of parameters kept fixed as given by Eq. (3) causes extinction in the top predator and then the system lose the persistence.
5. Increasing the parameter r_{14} ($r_{14} \geq 0.26$) with the rest of parameters kept fixed as given by Eq. (3) causes extinction in the top predator and then the system lose the persistence. However, decreasing the parameter r_{14} ($r_{14} \leq 0.15$) with the rest of parameters kept fixed as given by Eq. (3) causes extinction in the intermediate predators (susceptible and infected) and then the system lose the persistence.

Consequently, the parameters (r_2, r_6, r_{11}, r_{12} and r_{13}), which stand for attach rate of top predator to the susceptible intermediate predator, death rate of intermediate predators, conversion rates of top predators respectively represent a bifurcation parameters of system (2), in fact each of them has only one bifurcation point. However, the death rate of top predator represents a bifurcation parameter of system (2) with two points of bifurcations.

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